Teach Every Student to Think Like a Mathematician

Grace Kelemanik

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@GraceKelemanik

FOSTERING MATH PRACTICES

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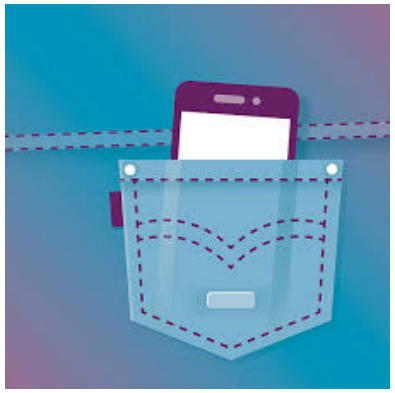
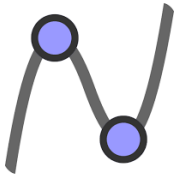
We are currently preparing students for jobs that don't yet exist . . . using technologies that haven't yet been invented . . . in order to solve problems we don't even know are problems yet.

—*The Jobs Revolution* Richard Riley






desmos

Students need ways into and through prickly problems

We must teach them to think like mathematicians!

Standards for Mathematical Practice

- **MP1** Make Sense of Problems and Persevere in Solving Them
- **MP2** Reason Abstractly and Quantitatively
- **MP3** Construct Viable Arguments and Critique the Reasoning of Others
- **MP4** Model with Mathematics
- **MP5** Use Appropriate of Tools Strategically
- **MP6** Precision in Mathematics
- **MP7** Look for and Make Use of Structure
- **MP8** Look for and Express Regularity in Repeated Reasoning

Support ALL Learners in multiple processing areas

Language

Conceptual

Visual – Spatial

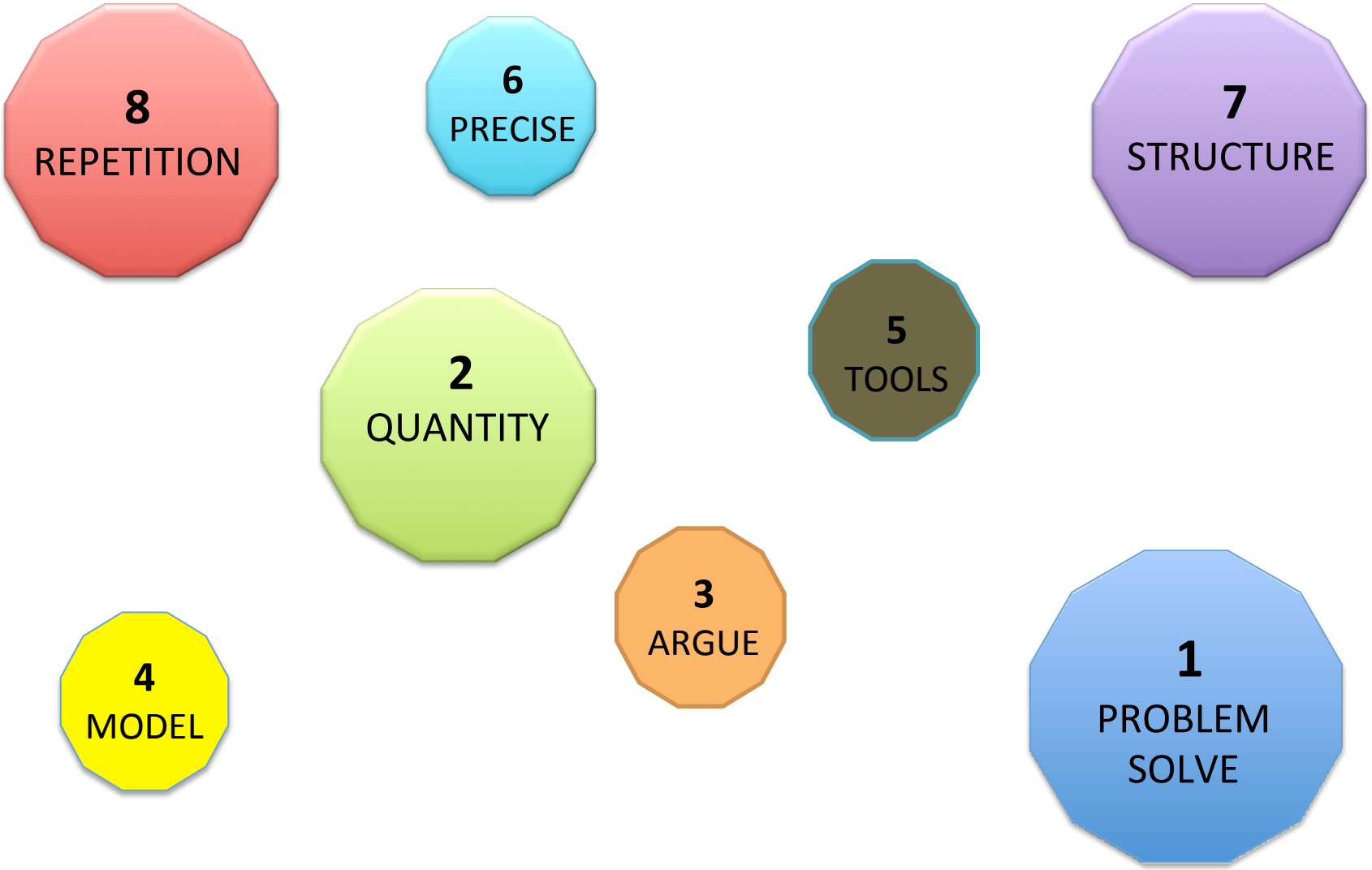
Attention

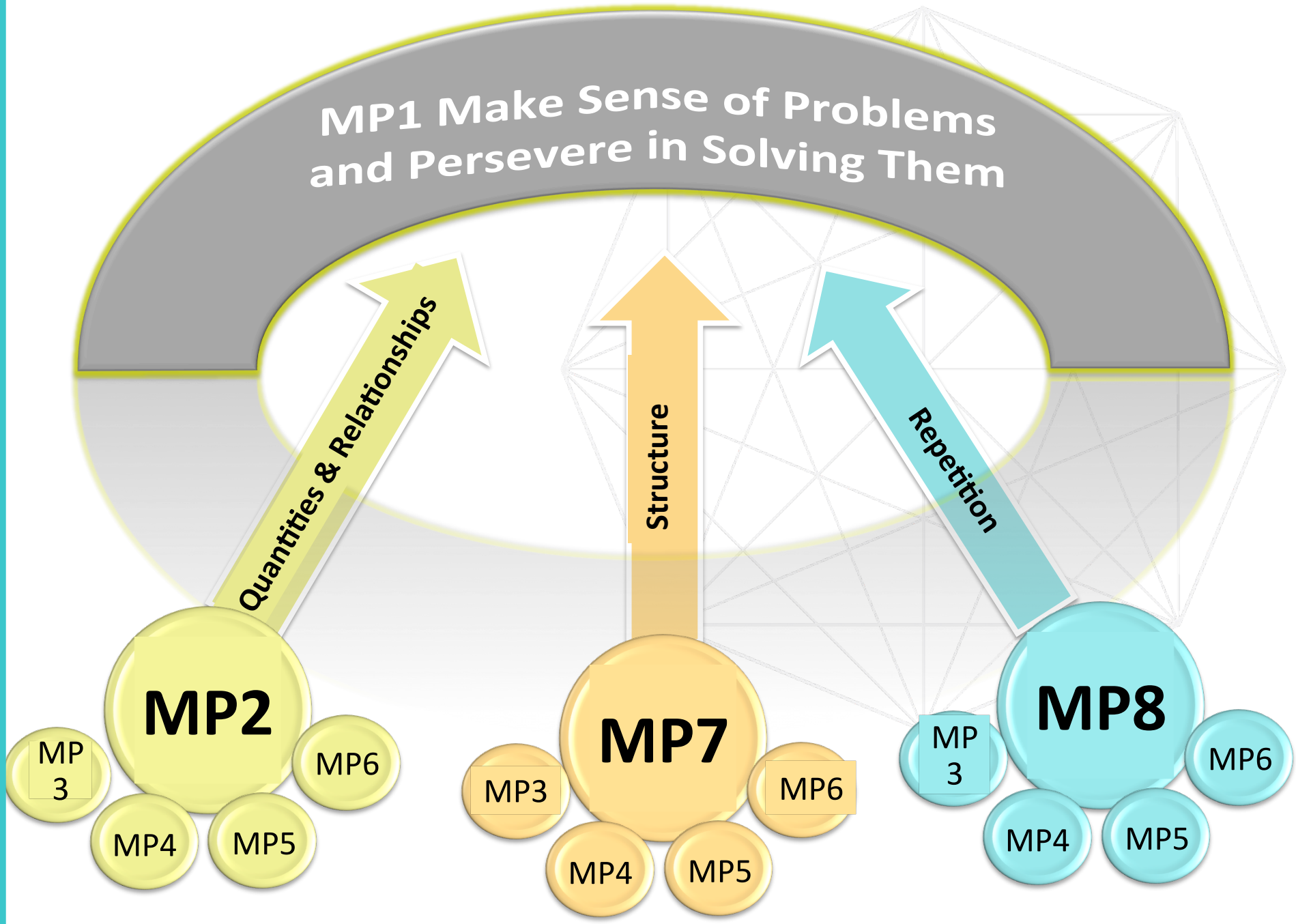
Organization

Memory


MP 1
MP 2
MP 3
MP 4
MP 5
MP 6
MP 7
MP 8

Not All Math Math Practices Are Equal!




MP1 Make Sense of Problems and Persevere in Solving Them
Quantities & Relationships
Structure
Repetition
MP2
MP 3
MP4
MP5
MP6
MP7
MP3
MP4
MP5
MP6
MP8
MP 3
MP4
MP5
MP6

Three Avenues of Thinking in Math

Quantitative Reasoning

Repeated Reasoning

Structural Thinking

Avenues of Thinking

Quantitative Reasoning	Structural Thinking	Repeated Reasoning

Attend to → Ask yourself → Actions

Fish Tank

Drill Down Experience

Shift

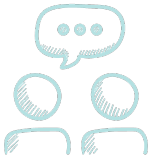
Avenue of Thinking for ALL learners

Fish Tank

A 20.5 gallon fish tank is 4/5 full. How many more gallons will it take to fill the tank?

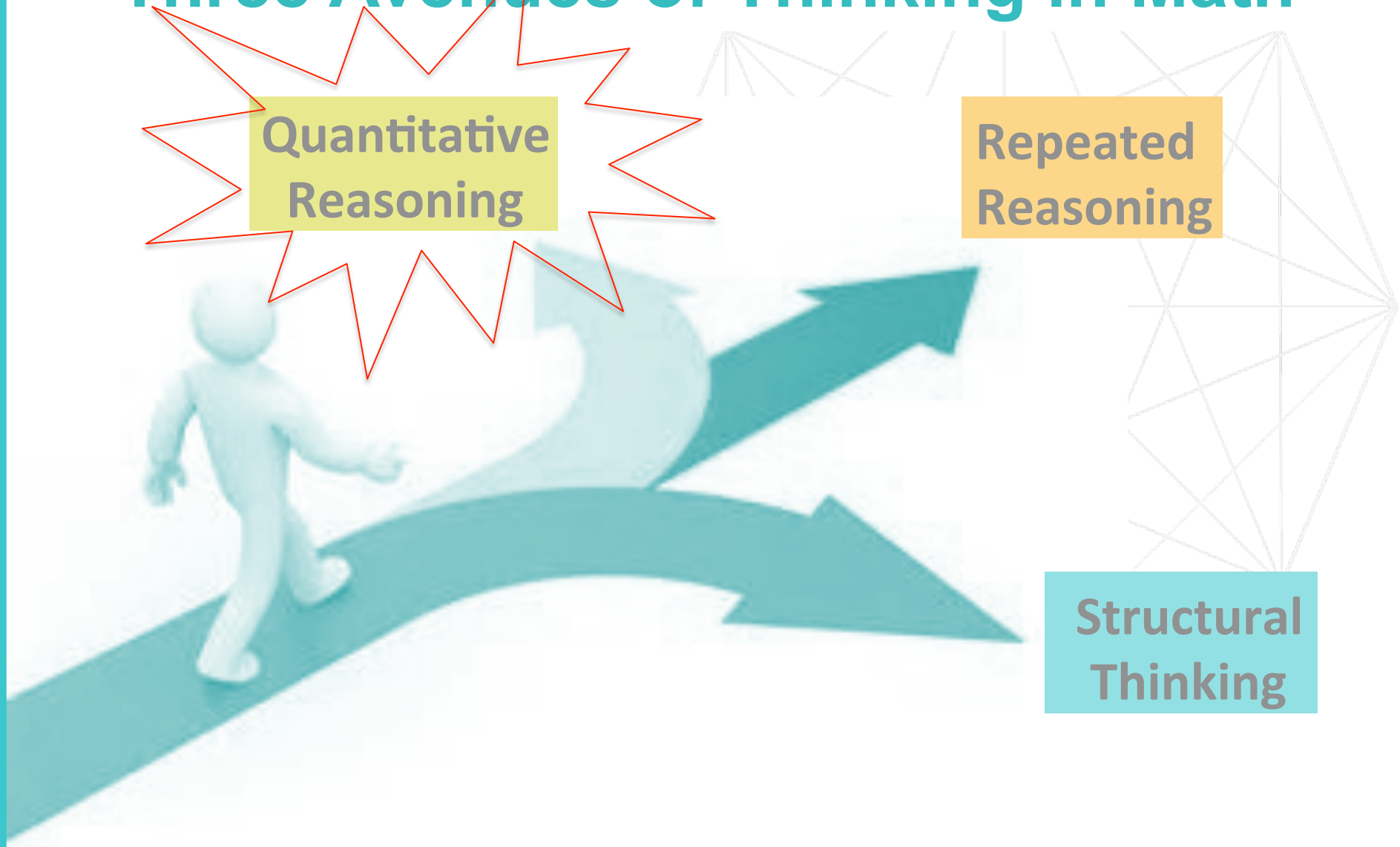
Instructions:

Solve the task on your own.



Discuss your approach with a partner.

Three Avenues of Thinking in Math



Quantitative Reasoning Avenue of Thinking

Attend to…

Quantities

and

Relationships

Ask yourself…

- What can I count or measure in this problem situation?
- How do the quantities relate to each other?
- How can I represent this problem?
- What does this (variable, number, shaded region, etc.) represent in the problem context?

Students typically attend to number.

7

1/2

Does the number tell me something about a **quantity** or is it describing a **relationship**?

Quantity or Relationship?

- Grace has 7 cookies
- Amy has 7 more cookies than Grace

- Grace ran ½ mile
- Amy ran ½ as far as Grace

What’s a Quantity?

- A Quantity is something you can count or measure
  - The number of…
  - The amount of...
- It answers the questions:
  - How many?
  - How much?

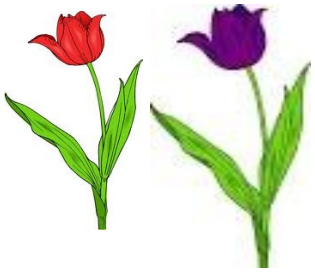
What's a Quantity?

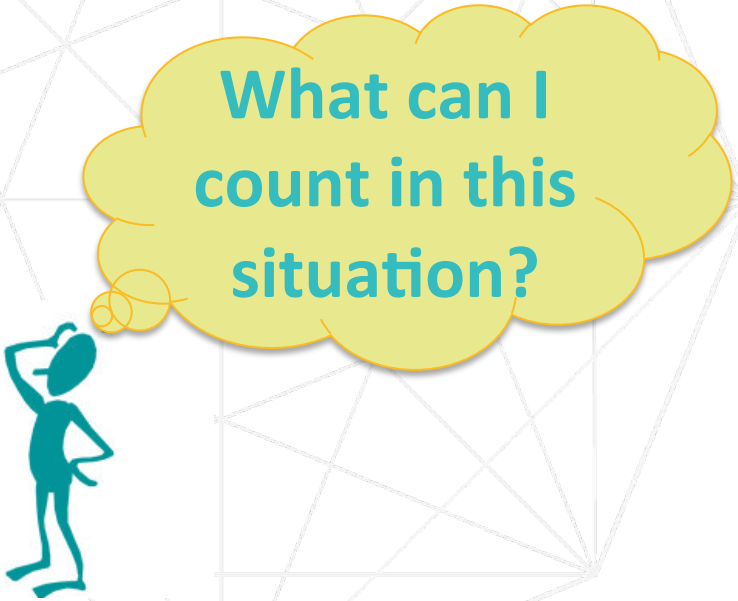
A Quantity has three parts…

- Value (e.g. 7 or ½ or 2π or ✗)
- Unit (e.g. cookies or miles)
- Sign (+ / -)

What are the Quantities in Gina's Garden?

Gina planted 24 flowers in her yard. Some of them were red and some of them were purple. There are twice as many purple flowers as red flowers.





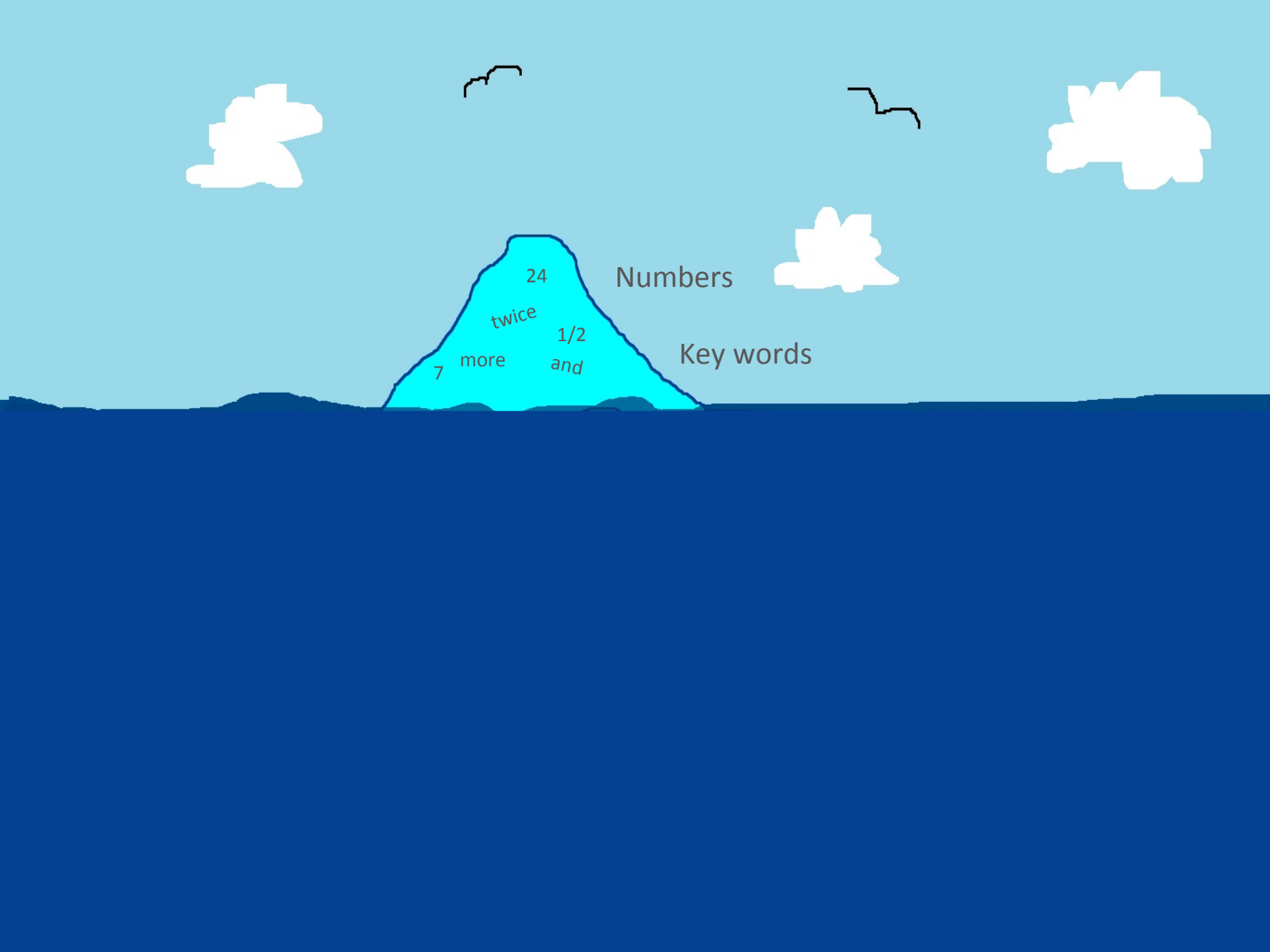
The number of…

QUANTITIES in Gina's Garden

I can count…

- The total number of flowers
- The number of red flowers
- The number of purple flowers
- The number of flowers that are neither red nor purple
- The number of colors of flowers

Quantities. In real world problems, the answers are usually not numbers but quantities: numbers with units, which involves measurement. In their work in measurement up through Grade 8, students primarily measure commonly used attributes such as length, area, and volume. In high school, students encounter a wider variety of units in modeling, e.g., acceleration, currency conversions, derived quantities such as person-hours and heating degree days, social science rates such as per-capita income, and rates in everyday life such as points scored per game or batting averages. They also encounter novel situations in which they themselves must conceive the attributes of interest. For example, to find a good measure of overall highway safety, they might propose measures such as fatalities per year, fatalities per year per driver, or fatalities per vehicle-mile traveled. Such a conceptual process is sometimes called quantification. Quantification is important for science, as when surface area suddenly "stands out" as an important variable in evaporation. Quantification is also important for companies, which must conceptualize relevant attributes and create or choose suitable measures for them.


24
twice
1/2
7
more
and
Numbers
Key words

Entering the Fish Tank via the Quantitative Avenue of Thinking

Pay attention to…

- The amount of water:
  - The tank holds
  - Already in the tank
  - Added to the tank

- Relationships between the amount of water the tank holds and the amount already in the tank

A 20.5 gallon fish tank is 4/5 full. How many more gallons will it take to fill the tank?

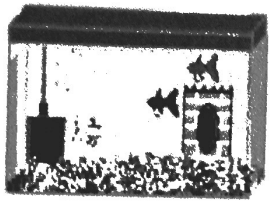
Entering the Fish Tank via the Quantitative Avenue of Thinking

Ask yourself…

- Is the 20.5 a value for a quantity or does it describe a relationship?
- What quantity has a value of 20.5?
- Is the 4/5 describing a relationship or a quantity?
- What two quantities have a 4/5 relationship?
- Are there quantities here that don't have a value given?
- How can I represent the quantities so I can see the relationship between them?

A 20.5 gallon fish tank is 4/5 full. How many more gallons will it take to fill the tank?

Quantitative Reasoning Avenue of Thinking

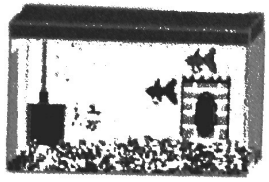


A 20.5 gallon fish tank is 4/5 full.
How many gallons will it take to fill the tank?

- [x] # of gallons in tank
- # of gallons tank holds (20.5)
- [ ] # of gallons needed to fill tank.

- # of gallons in TANK is 4/5 of The # of gallons The TANK holds

Quantitative Reasoning Avenue of Thinking



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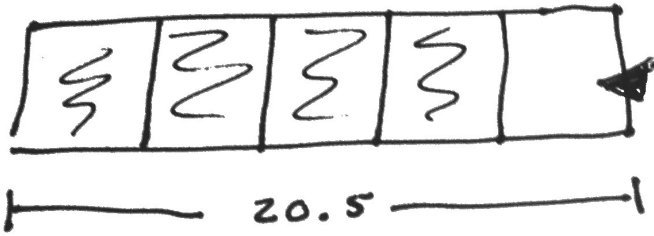


20.5 ÷ 5

4.1

Quantitative Reasoning Actions

- Identify quantities explicitly mentioned in the problem statement
- Surface hidden or implied quantities
- Note relationships between quantities
- Abstract problem situations
- Use representations to see quantities and relationships
- Recall and consider referents

When students only attend to numbers…

A 20.5 gallon fish tank is 4/5 full.
How many gallons will it take to fill the tank?

$$\frac{4}{5} \times 20.5$$

$$\frac{\cancel{4}^{2}}{\cancel{5}_{1}} \times \frac{\cancel{205}^{41}}{\cancel{10}_{5}} = \frac{82}{5} = 16\frac{2}{5}$$

$$20\frac{5}{10} = \frac{205}{10}$$

$$5\overline{)82} \quad 16$$
$$5$$
$$32$$
$$30$$
$$2$$

Quantitative Reasoning Shifts

Look beyond the numbers and key words in a problem statement

To the quantities and relationships those numbers and key words describe

How could quantitative reasoning support students?

Number Grabbing → Understanding Quantities

Blind Operating → Working with relationships

Quantitative Reasoning supports ALL students….especially

- Students who don’t know where to begin to solve a word problem
- Students who struggle with multi-step problems
- Students who benefit from working within contexts
- Students who benefit from drawing/using visual representations

Three Avenues of Thinking in Math

Quantitative Reasoning

Repeated Reasoning

Structural Thinking

Structural Reasoning
Avenue of Thinking

Attend to…

Organization

and

Properties

of

Number and

Space

Ask yourself…

- Is this behaving like something else I know?
- How can I use properties to uncover structure?
- How can I change the form to make it easier to work with?
- How can I "chunk" this to make sense of it?
- How can I connect this to math I know?

Entering the Fish Tank via the Structural Avenue of Thinking

Pay attention to…

- Types of numbers
- Composition of fractions

A 20.5 gallon fish tank is 4/5 full. How many more gallons will it take to fill the tank?

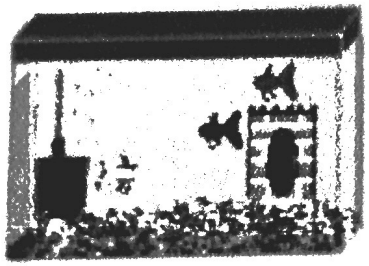
Entering the Fish Tank via the Structural Avenue of Thinking

Ask yourself…

- Is there another way I can think about 4/5 full?
- How can I change the form of 20.5 and 4/5 to make them easier to work with?

A 20.5 gallon fish tank is 4/5 full. How many more gallons will it take to fill the tank?

Entering the Fish Tank via the Structural Avenue of Thinking

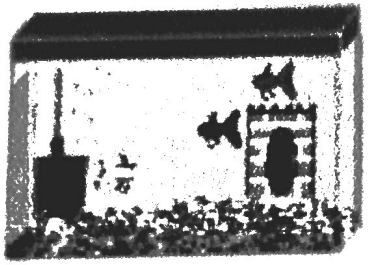


A 20.5 gallon fish tank is 4/5 full. — 1/5 Empty

How many gallons will it take to fill the tank?

$\frac{1}{5}$ of 20.5

Entering the Fish Tank via the Structural Avenue of Thinking



A 20.5 gallon fish tank is 4/5 full.
How many gallons will it take to fill the tank?

1/5 EMPTY

$\frac{1}{5}$ of 20.5

$\frac{1}{5}(20) = 4$

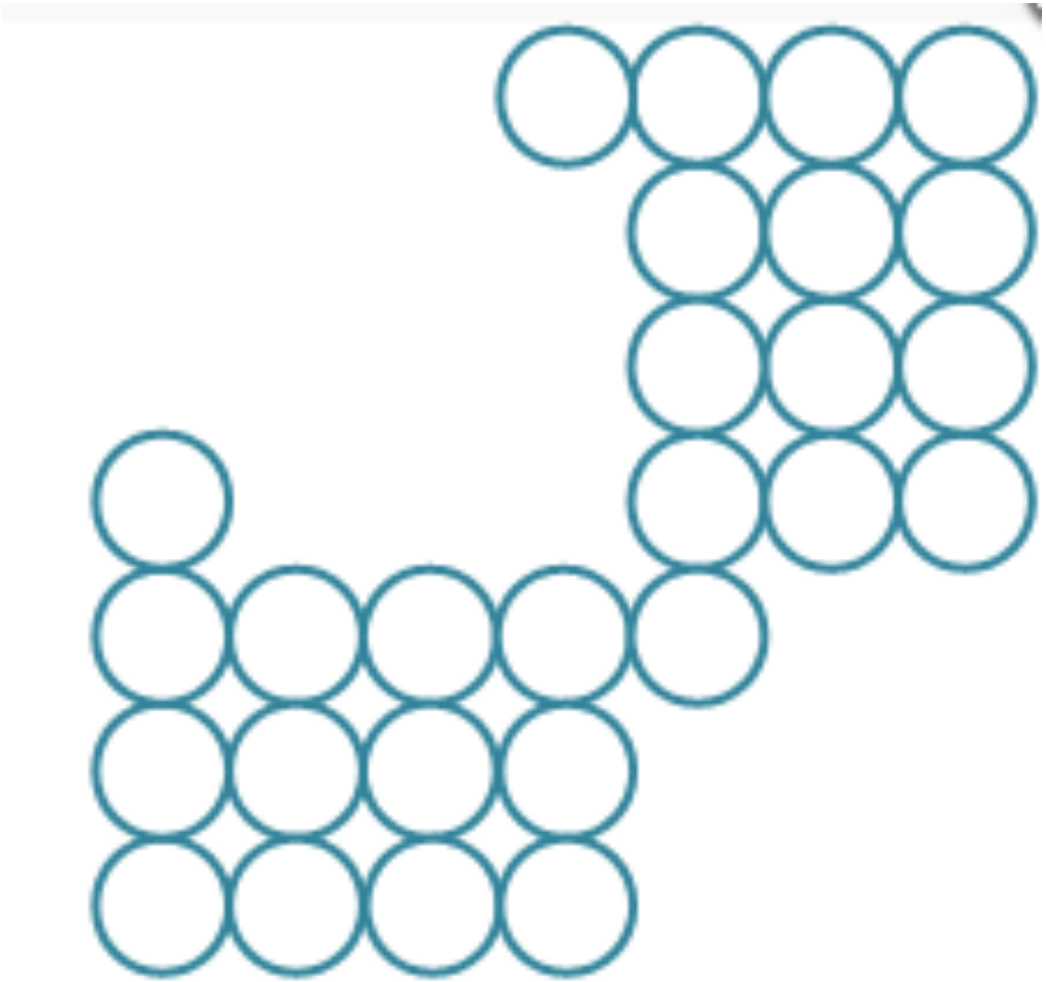
$\frac{1}{5}(.5) = .1$

$\boxed{4.1 \text{ gal.}}$

Structural Thinking Actions

- *Chunk* complicated objects
- *Connect* math ideas and representations
- *Change* the form of objects
- Recall and use properties, rules of operations, and geometric relationships
- Shift perspective

Find the total number of circles quickly "in your head" (i.e. without counting every single circle)



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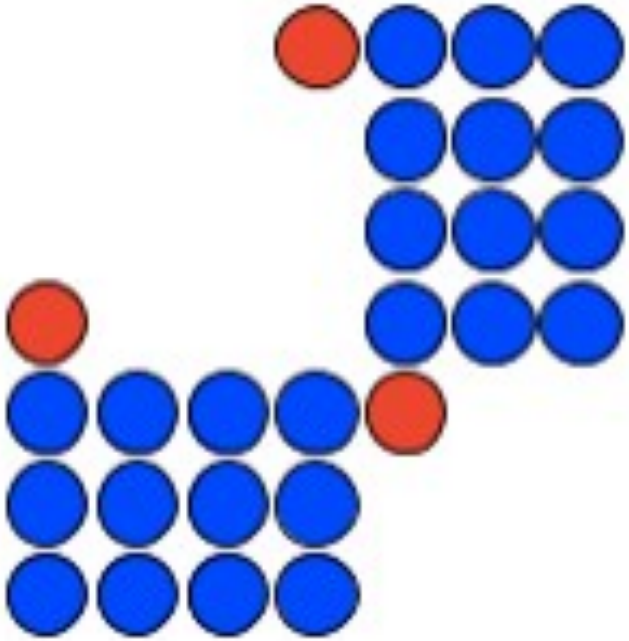
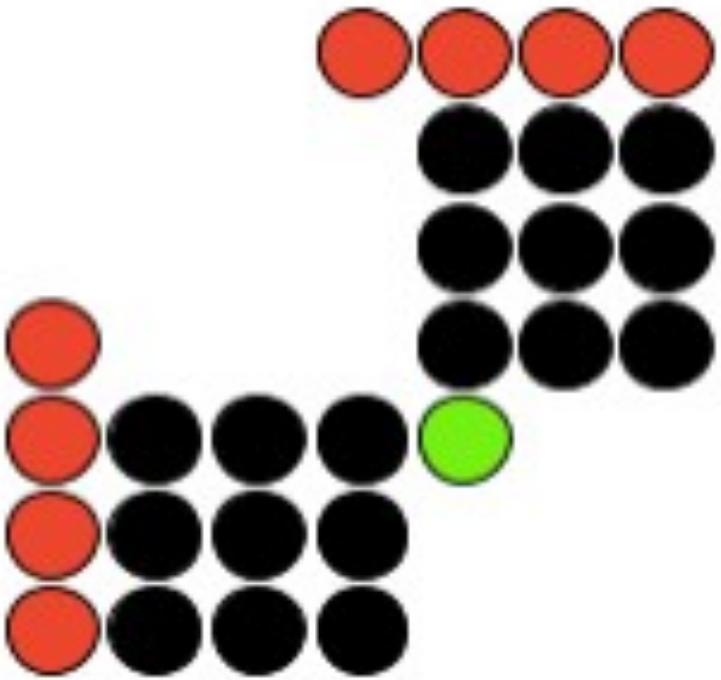
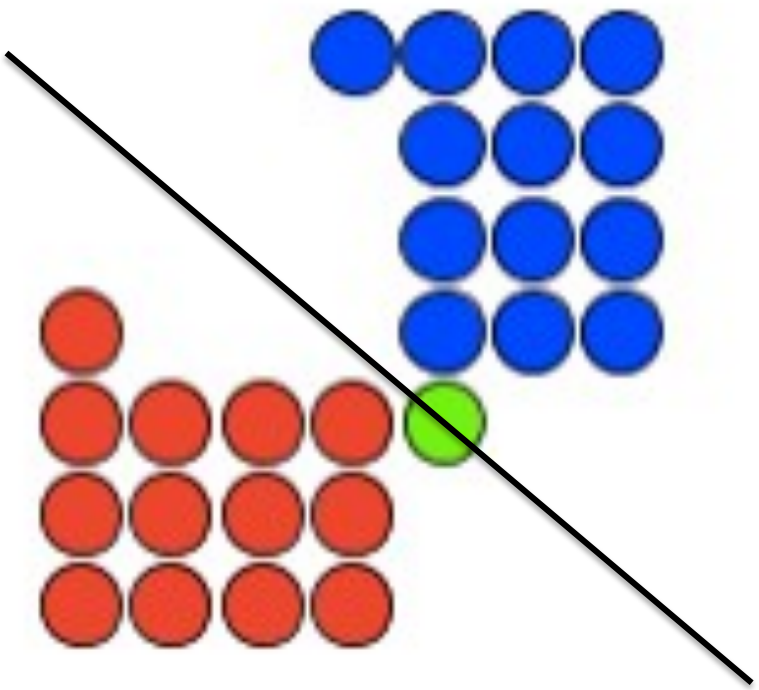
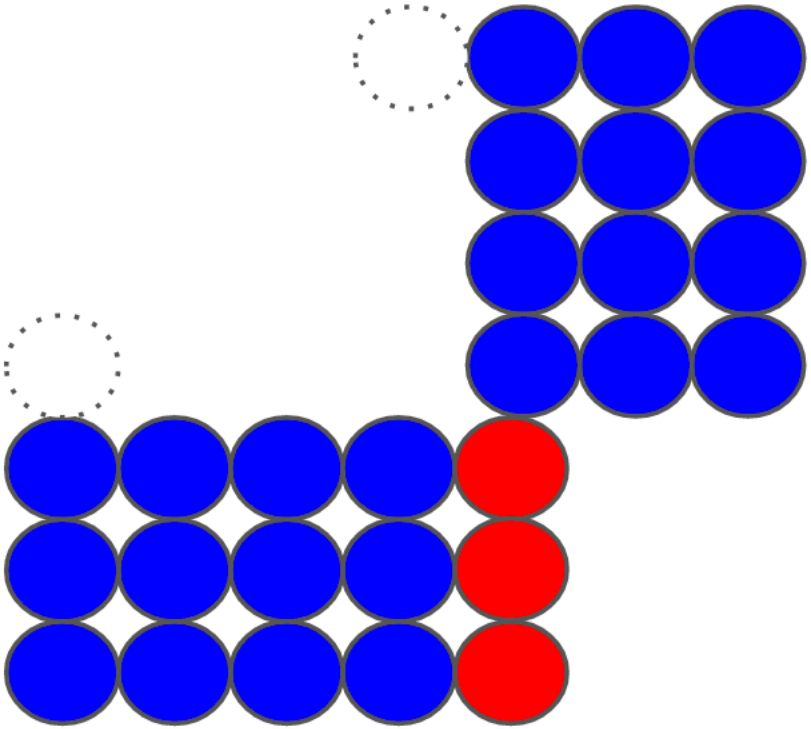
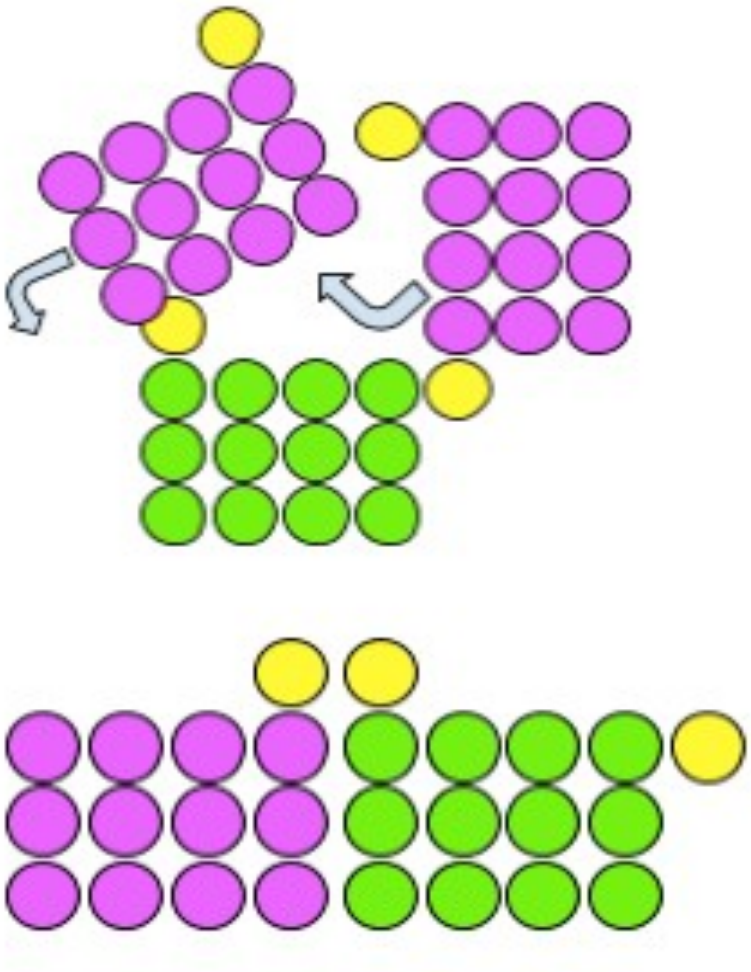
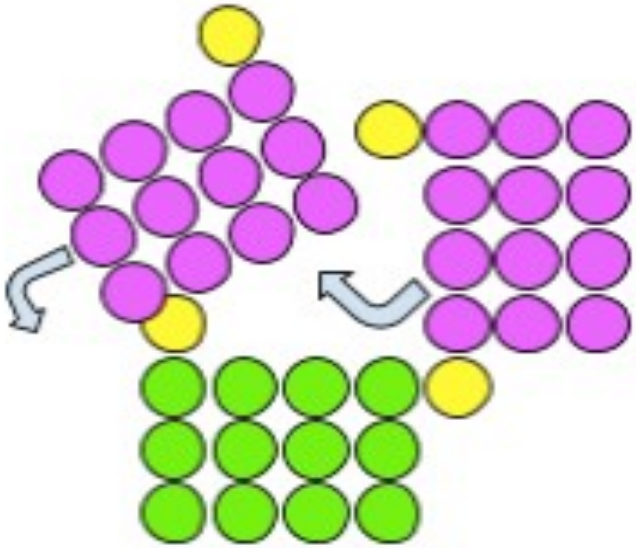


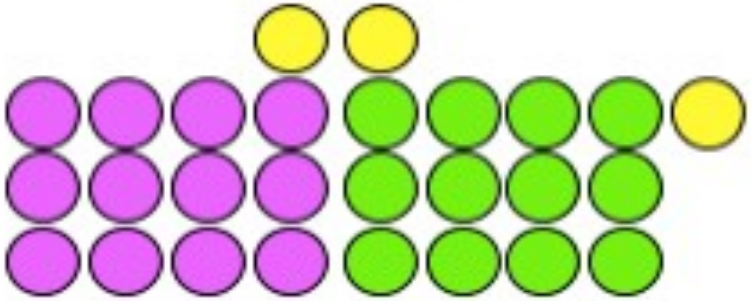


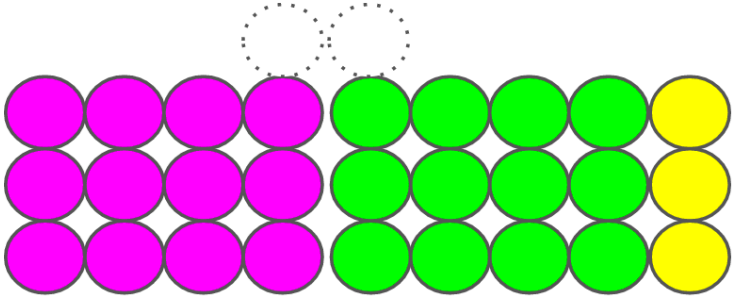




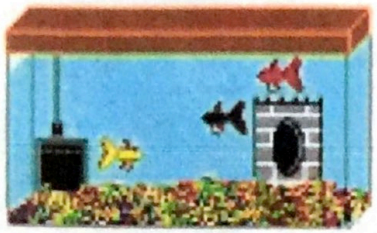








When students don't think structurally…



A 20.5 gallon fish tank is 4/5 full.
How many gallons will it take to fill the tank?

$\frac{4}{5} \times 20.5$

$\frac{4}{5} \times 20\frac{5}{10}$

$\frac{4}{5} \times \frac{205}{10} = \frac{100}{50} = 2$

$$\begin{array}{r} 205 \\ \times\ 4 \\ \hline 100 \end{array}$$

$$\begin{array}{r} 20.5 \\ -\ 2.0 \\ \hline 18.5 \end{array}$$

Structural Thinking Shifts

A collection of unrelated results and procedures to know

A set of interconnected ideas that build on each other and make sense

Structural Thinking supports ALL students….especially

- Students who lose track of their work and/or calculations
- Students who see the ‘big picture’
- Students who benefit from multiple representations

Three Avenues of Thinking in Math

Quantitative Reasoning

Repeated Reasoning

Structural Thinking

Repeated Reasoning (MP8)

Attend to...

Repetition in
Processes

Counting

Calculating

Constructing

Ask yourself...

- Do I keep doing the same thing over and over again?
- What about the process is repeating?
- How can I generalize the repetition?
- Have I included every step?

Repeated Reasoning (MP8)

Actions You Take…

- Count in an organized way
- Draw or build several figures
- Try several numbers and observe the process
- Record and track calculations
- Generalize the repetition
- Simultaneously maintain oversight of the process while attending to details
- Monitor and evaluate reasonableness of intermediate results

Decompressing Repeated Reasoning

- Pay attention to the process
- Sense the regularity
- “Shortcut” the process
- Connect the process to an “input” value
- Generalize the process to a rule

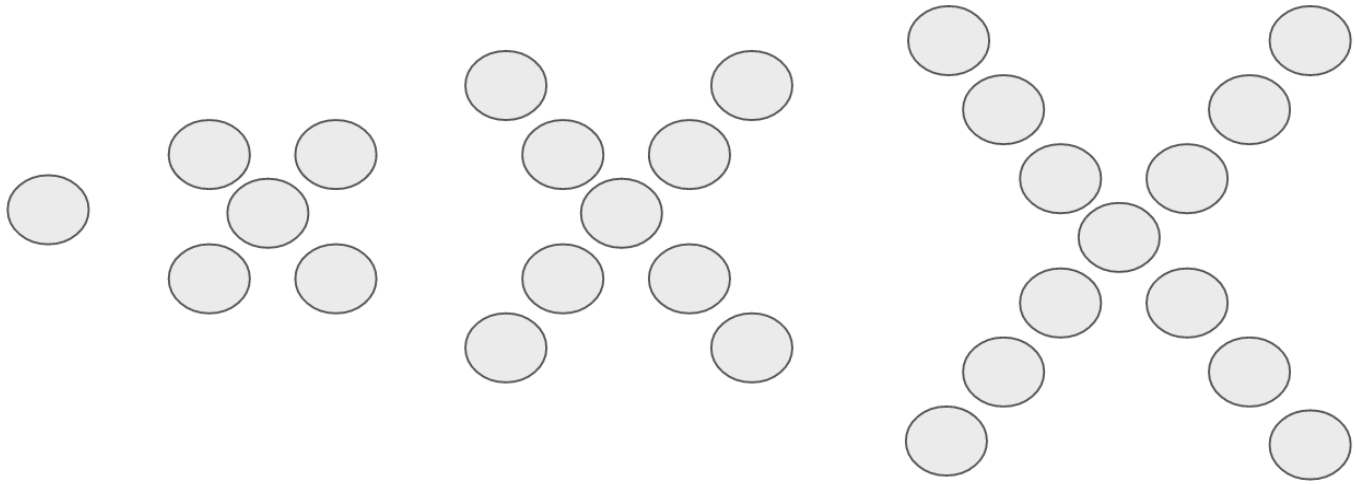
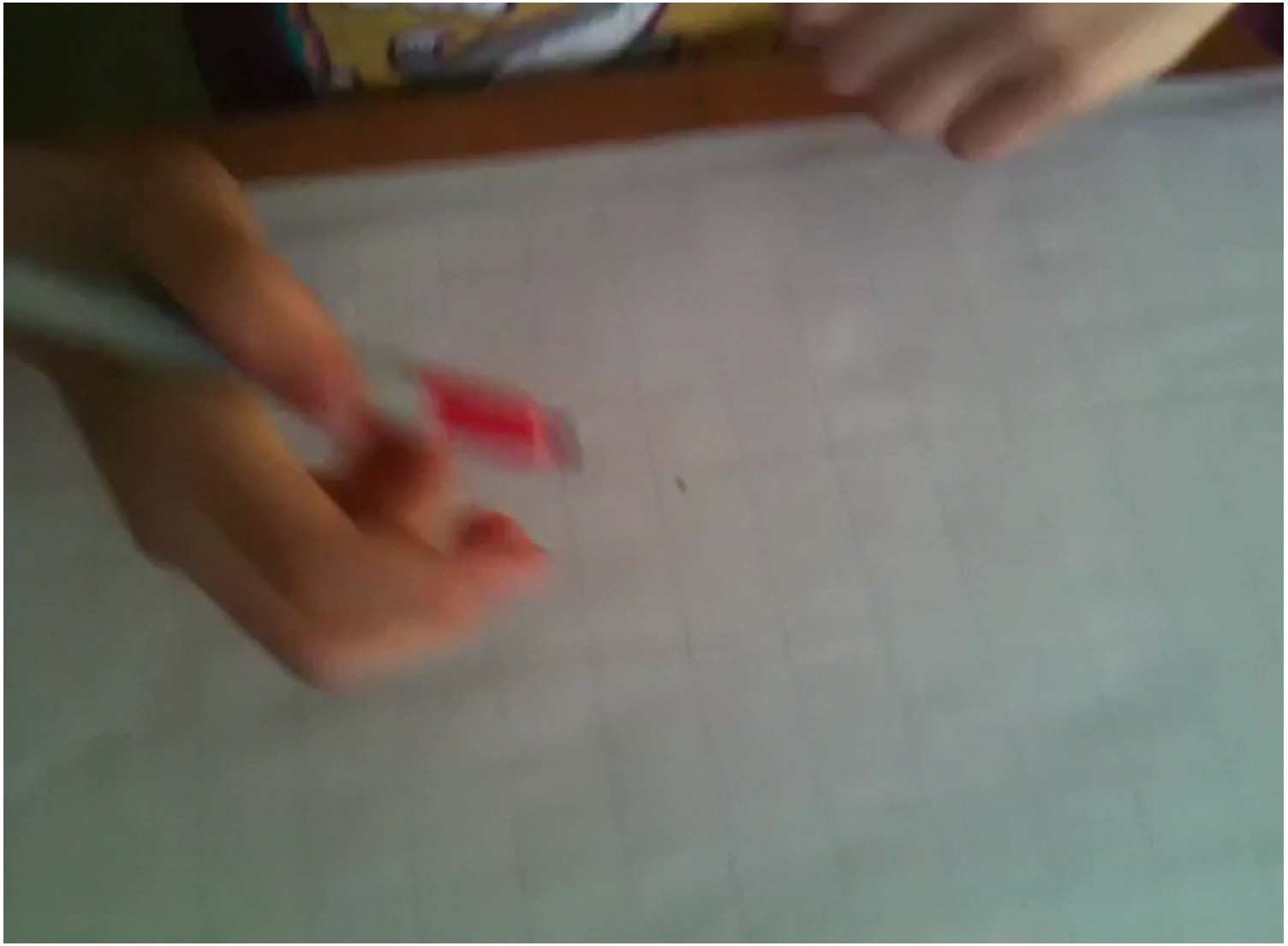


Figure 1 Figure 2 Figure 3 Figure 4

Figure #	1	2	3	4	5	10	N
# of Dots	1	5	9	13			

Repetition in Constructing



Repetition in Constructing

Decompressing Repeated Reasoning

- Pay attention to the process
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- Generalize the process to a rule

Repeated Reasoning (MP8)

Attend to Repetition in Processes

Counting

Constructing

Calculating

Repeated Reasoning (MP8)

Adam has a 20.5 gallon fish tank that is ⅘ full. How many gallons will it take to fill the tank?

10 gallons?

$10 + (10 + 10 + 10 + 10) \neq 20.5$

$5 \times 10 \stackrel{?}{=} 20.5$

5 gallons?

$5 + (5 + 5 + 5 + 5) \stackrel{?}{=} 20.5$

$5 \times 5 \stackrel{?}{=} 20.5$

2 gallons?

$2 + 2(4) \stackrel{?}{=} 20.5$

$5 \times 2 \stackrel{?}{=} 20.5$

Repeated Reasoning (MP8)

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$5 \times 2 \stackrel{?}{=} 20.5$

$5 \cdot \square = 20.5$

$\frac{20.5}{5} = 4.1$

When students focus on number pattern vs. repetition in process

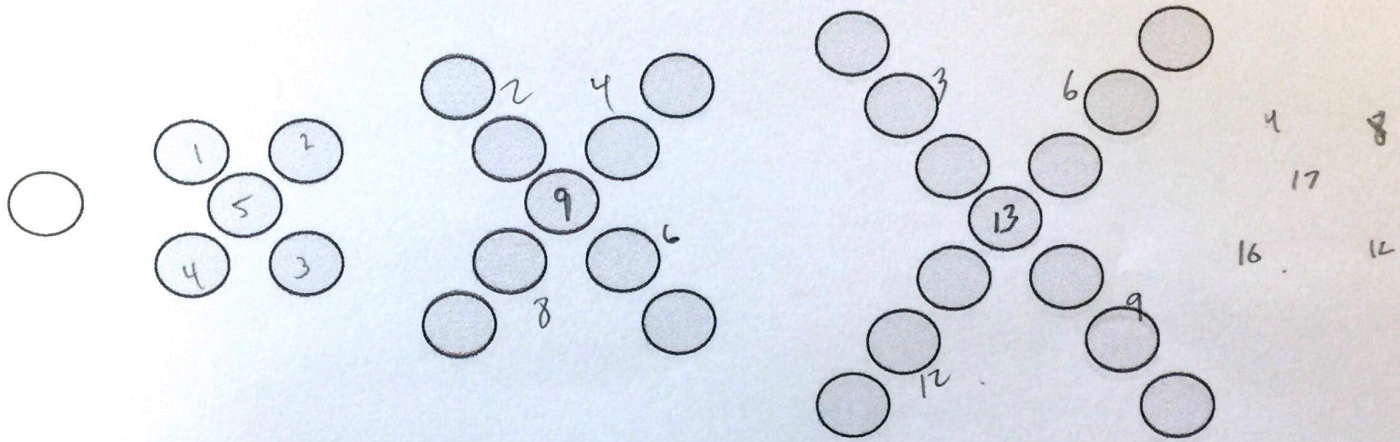

Figure 1 Figure 2 Figure 3 Figure 4

Figure #	1	2	3	4	5	10	100	N
# of Dots	1	5	9	13	17	40	400	4N

4 4 4 4

Find a rule to determine the number of circles in any figure?

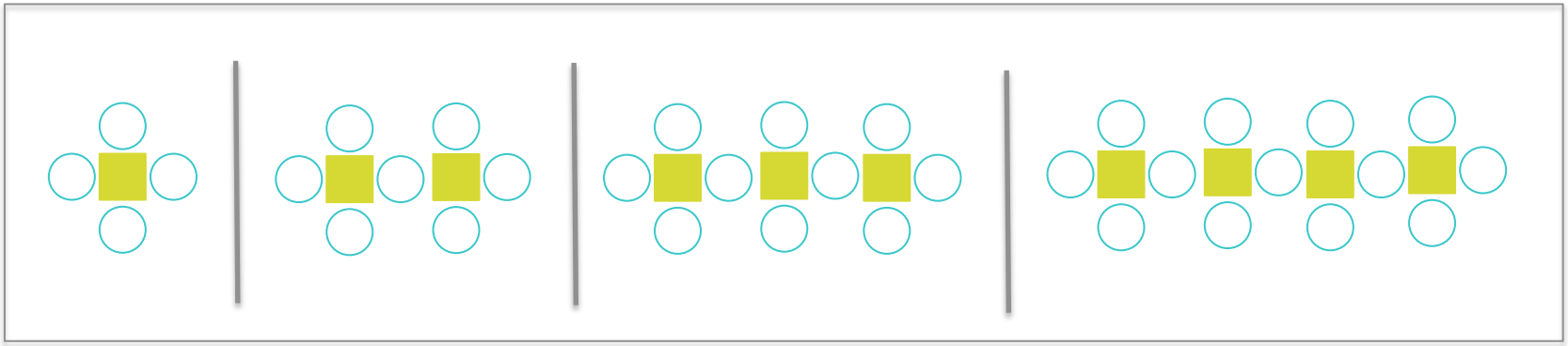

Figure 1 Figure 2 Figure 3 Figure 4

Figure #	1	2	3	4	10	100	N
# of Circles							

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Find a rule to determine the number of circles in any figure?

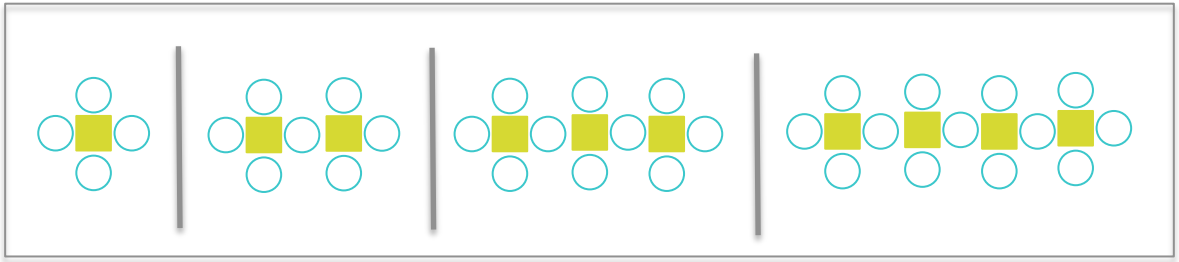

Figure 1 Figure 2 Figure 3 Figure 4

Figure #	PROCESS Counting, Calculating, Constructing	# of Circles
1		
2		
3		
4		
10		
100		
N		

http://www.visualpatterns.org/

WE HAVE TO PRIVILEGE THE PROCESS!

Repeated Reasoning Shifts

Patterns in Numbers and Results

→

The counting, calculating and constructing processes that generated those numbers and results

How could repeated reasoning support students?

'Magic' Rules

Generalizations that are rooted in concrete processes

Repeated Reasoning supports ALL students….especially

- Students who benefit from multiple modalities
- Students who struggle to abstract and generalize
- Students who work in organized and/or systematic ways
- Students who benefit from seeing how rules are developed

Teach the Avenues of Thinking!

Quantitative Reasoning

Repeated Reasoning

Structural Thinking

For More on Instructional Routines

Reach Out

GraceKelemanik@gmail.com

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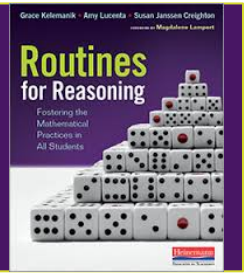
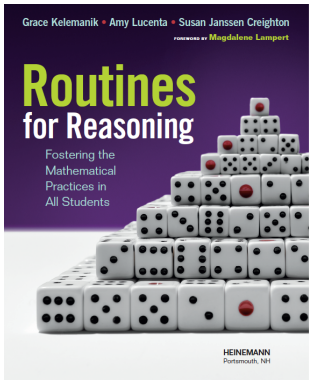


Make it Routine!

- Mathematical thinking is a habit.
- Habits of thinking are formed through routine
- Use Instructional Routines to develop the Avenues of Thinking!



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“Teaching students to think and reason is perhaps the greatest challenge we face as math educators, and these routines provide clear pathways to do so.”

Starting the Year with Contemplate then Calculate

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FRI 15 SEP 2017 - SAT 16 SEP 2017

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